Introduction to Public Key Cryptography and Modular Arithmetic.

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1 Introduction.

These notes are an introduction to the RSA algorithm, and to the mathematics needed to understand it. The RSA algorithm — the name comes from the initials of its inventors, Rivest, Shamir, and Adleman — is the foundation of modern public key cryptography. It is used for electronic commerce and many other types of secure communication over the Internet.

The RSA algorithm is based on a type of mathematics known as modular arithmetic. Modular arithmetic is an interesting variation of ordinary arithmetic, but whereas everyday arithmetic is familiar to school children everywhere, modular arithmetic is a somewhat obscure subject. It's not that modular arithmetic is particularly difficult, or confusing; one could teach it in high school, or even earlier. Conventional thinking, however, places a higher value on the ability to balance a checkbook than on the ability to communicate in code; this is probably the reason why most people have never heard of modular arithmetic. Today, the rapid proliferation of the Internet and the growing popularity of electronic financial transactions are causing a shift in attitudes. These days, even an introductory understanding of public key cryptography can be enormously useful. Consequently, today's students deserve an opportunity to become acquainted with the methods and ideas of modular arithmetic.

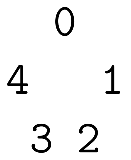
First, a quick word about these notes. I have tried to make the material here as down to earth, and accessible as possible. As such, the emphasis is on concrete calculations, rather than abstruse theory. My goal is to guide you through the concrete steps needed to implement and understand RSA encryption/decryption. Therefore in the present context, calculations are in and abstraction is out. A number of simple exercises is included; you should work them as you go along. The answers to the exercises are collected in a final appendix.

So set aside a few hours, go grab a pencil, and pour yourself a cup of coffee. It shouldn't take very long to go through these notes; the material in here just isn't all that hard.

2 Modular Arithmetic

I remember that as a child, after I learned the general system of counting, I was fascinated by and frequently thought about the fact that *numbers never end.* In principle, one can count as high as one wants — an activity I experimented with as a child, and one that serves me well in adulthood when insomnia comes calling. What, however, would be the consequences if numbers *did end*? What if numbers behaved like the numbers on a clock? What if, as one counted higher and higher, one would come full circle and begin counting again from the beginning, from zero?

To make this idea concrete suppose that there are only five different numbers: $0, 1, 2, 3, 4$; and that these numbers are arranged on a circular clock in the usual clock-wise direction, as in the figure below. It's not hard to imagine how to do addition using this alternate system of counting.



Doing $2 + 2$ one starts with 2, increments twice, and obtains the answer 4. Doing $2 + 4$, however, is a little unusual, because this time the answer is 1. The reason is that after 3 steps one comes full circle, i.e. to 0; from there, one more increment brings the final total to 1. Subtraction isn't hard to understand either; one counts backwards, i.e. counter-clockwise, rather than forwards. So, in the clock-like way of counting, $2 - 4 = 3$; one starts with 2 and turns the clock back 4 times.

The usual name for this kind of counting is modular arithmetic (some people have also called it "clock arithmetic".) It is important to note that there are many different modular arithmetics; it all depends on how big the clock is. The size of the clock, i.e. the total of all possible numbers, is called the *modulus*. In the example of the preceding paragraph the modulus was 5. One can just as happily work with other moduli: 7, 10, 123; any integer greater than 1 will do. For example, the binary arithmetic of ones and zeroes that underlies the workings of modern computers is nothing but arithmetic with a modulus of 2.

In addition to the clock analogy there is another explanation of modular arithmetic that needs to be mentioned. This alternate system is made up of all the possible integers; however any two numbers that generate the same displacement on a clock are considered to be equivalent. Thus modulo 5 the numbers $-8, -3, 2, 7, 12, 17, 21$ are all considered to be equivalent, because they all correspond to a displacement of 2 spaces on a 5-space clock. To put it another way, two numbers are considered equivalent if and only if they differ by a multiple of 5. The mathematical notation for this equivalence works like this. One writes:

$$6 \equiv 16 \pmod 5,$$

and says out loud: "six is equivalent to sixteen modulo five". The meaning of this statement is that 6 and 16 differ by a multiple of 5; or, equivalently that 6 and 16 generate an equivalent displacement on a clock with 5 spaces. In summary, in the modulo 5 system, there are exactly five classes of numbers: the numbers equivalent to 0, the numbers equivalent to 1, to 2, to 3, and to 4.

Exercise 1 *Which of the following statements are true, and which are false?*

$$1{-}8 \equiv 0 \pmod 7, \quad 1{-}8 \equiv 3 \pmod 5, \quad 9{+}9{+}9 \equiv 8 \pmod{13}, \quad 2{+}11{-}33 \equiv 0 \pmod{10},$$

In a system of modular arithmetic with modulus n it is possible to reduce every integer to an equivalent number between 0 and $n - 1$. To do the reduction one divides and calculates the remainder. For example, let's say one wants to calculate the reduced form of 2040 modulo 209. One has

$$2040 \div 209 = 9\tfrac{159}{209}, \quad \text{i.e. } 2040 = 9 \times 209 + 159.$$

Therefore $2040 \equiv 159 \pmod{209}$. Again, the meaning of this statement is that a total displacement of 2040 spaces on a clock with 209 places yields a net displacement of 159 spaces.

By the way, save yourself some grief and check the documentation of that expensive calculator you bought when you came to university. Many modern calculators have a built-in remainder function, and possibly other functions for performing the operations of modular arithmetic.

2.1 Multiplication

As everyone knows, multiplication is nothing but repeated addition. This is the meaning of multiplication in everyday arithmetic; this is how multiplication works in modular arithmetic as well. Say one wants to calculate 2×3, i.e. $2 + 2 + 2$, in the modulo 5 system. The answer of course is 6, which modulo 5 is equivalent to 1. Writing this more succinctly: $2 \times 3 \equiv 1 \pmod 5$.

How sensible is the above definition of multiplication? Does it obey the same algebraic laws as ordinary multiplication? The answer is, yes! For example, ordinary multiplication obeys a rule called the distributive law. As a particular instance of this rule, $(3+4) \times 2 = 7 \times 2$ is equal to $3 \times 2 + 4 \times 2 = 6 + 8$.

What about modular arithmetic; does the distributive law continue to work? Consider the last calculations modulo 5. For the first expression one gets

$$(3 + 4) \times 2 = 7 \times 2 \equiv 2 \times 2 \equiv 4 \pmod 5.$$

For the second expression one gets

$$3 \times 2 + 4 \times 2 = 6 + 8 \equiv 1 + 3 \equiv 4 \pmod 5;$$

the two answers agree.

What about something like 8×2 versus 3×2? Since 3 and 8 are equivalent modulo 5, it stands to reason that the two answers should also be equivalent. This is indeed the case; one gets an answer of $1 \pmod 5$ for both calculations. The answers are the same because 8 and 3 differ by a multiple of 5; indeed, $8 = 3 + 5$. Therefore

$$2 \times 8 = 2 \times (3 + 5) = 2 \times 3 + 2 \times 5.$$

Now 2×5 is a multiple of 5 and is consequently equivalent to $0 \pmod 5$. Therefore

$$2 \times 8 = 2 \times 3 + 2 \times 5 \equiv 2 \times 3 + 0 \pmod 5.$$

In summary, multiplication is a perfectly valid, consistent operation in the world of modular arithmetic.

Exercise 2 *Complete the following table of multiplication modulo* 5.

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	.	.	.	.
4	.	.	.	.

A surprising aspect of modular arithmetic is the fact that one can also do division. Recall that every division operation in ordinary arithmetic can be recast as a corresponding multiplication by a reciprocal. Thus, $a \div b = a \times b^{-1}$, where b^{-1} is the one particular number such that $b \times b^{-1} = 1$. A moment's worth of reflection will reveal that this definition makes perfect sense for modular arithmetic. Say one wanted to find the reciprocal of 2 modulo 5. A consultation of the multiplication table from the preceding exercise readily yields the answer: $2 \times 3 \equiv 1 \pmod 5$, and therefore one writes $2^{-1} \equiv 3 \pmod 5$. In a sense, division in modular arithmetic is easier than it is in ordinary arithmetic; one doesn't have to worry about fractions. For example, division by 2 is equivalent to a multiplication by 3. Thus $3 \div 2 \equiv 3 \times 3 \equiv 4 \pmod 5$. One can even check the answer: $4 \times 2 \equiv 3 \pmod 5$; everything is correct.

Exercise 3 *Find the reciprocals of* 3 *and* 4 *modulo* 5*. Perform the following divisions modulo* 5 *and then check the answers by performing the necessary multiplications:* $4 \div 3$, $3 \div 4$, $3 \div 3$, $1 \div 3$.

2.2 It's a strange world, after all.

Who can forget that old bugaboo of elementary school arithmetic: the problem of division by zero? "Division by zero is ..." There are a number of typical endings to this sentence, and all of them serve to imply that division by zero is somehow bad, if not outright impossible. My preference in the face of such a question is to be as undogmatic as possible, and to pass the ball back to the questioner: "I don't know what one divided by zero is, my friend, but if you would like to hazard an answer, I will be happy to check it for you. What did you say? One divided by zero is three? I don't think so. You see, zero times three is zero, not one; so I'm afraid you shall have to try again." In this fashion the questioner should quickly become convinced that

$1 \div 0$, whatever such an answer may be, cannot be an ordinary, everyday number. So perhaps what one should really be saying is that the equation $? \times 0 = 1$ has no solutions.

Division by zero is just as problematic in modular arithmetic as it is in ordinary arithmetic. Furthermore, in the world of modular arithmetic, there exist, in addition to zero, other "division unfriendly" numbers. In order to see this, consider the following operation in the modulo 6 system: $1 \div 4$. In other words, try to solve the following equation: $4 \times ? \equiv 1 \pmod 6$. No such solution exists, of course. Think about the multiplication table modulo 6. The row that begins with 4 reads: $4, 2, 0, 4, 2$. In contrast to the way multiplication worked modulo 5, certain rows of the multiplication table modulo 6 have repeated entries, and consequently, in such rows, certain other numbers do not appear at all. So $1 \div 4$ cannot be given a value, because the number 1 doesn't occur in row number 4. Notice that something like $4 \div 4$ doesn't work either, but now ambiguity is to blame. There are two possible solutions to the equation $4 \times x \equiv 4 \pmod 6$; both $x = 1$ and $x = 4$ will work.

Exercise 4 *Write down the multiplication table modulo* 6*.*

These strange "division unfriendly" numbers possess another curious property. Notice that $4 \times 3 \equiv 0 \pmod 6$. One would never expect to see an equation like that in ordinary arithmetic; one simply doesn't expect to multiply two non-zero numbers and have the answer turn out to be zero. As unusual as this may appear at first glance, such goings on are quite commonplace in the world of modular arithmetic. In fact, there is some standard terminology that serves to describe such situations. If the product of two given numbers is zero, one calls these numbers *divisors of zero.* It's an apt title, because such numbers are literally able to divide zero and have a non-zero number be the answer. Modulo 6 the divisors of zero are 2, 3, and 4, because

$$2 \times 3 \equiv 4 \times 3 \equiv 0 \pmod 6.$$

What about the remaining numbers in the mod 6 system; what about 1 and 5? These are the "division friendly" numbers. Again there is some standard terminology that one should learn at this point. A number is called *a unit* if and only if it possesses a multiplicative reciprocal. Thus, both 1 and 5 have reciprocals and are therefore called units: $1^{-1} \equiv 1 \pmod 6$ and $5^{-1} \equiv 5 \pmod 6$

Exercise 5 *Explain why in modular arithmetic a unit can never be a divisor of zero, and why a divisor of zero can never be a unit.*

2.3 GCD: the Greatest Common Divisor

The acronym GCD stands for *greatest common divisor*. A common divisor of numbers a and b is a number that evenly divides both a and b. The greatest common divisor of a and b is simply the largest such common divisor. Consider for example the numbers 12 and 16. Obviously 2 is a common divisor of both. However, 4 is also a common divisor, and in fact it is the largest common divisor. Therefore 4 is the GCD of 12 and 16.

Note that 1 is a common divisor of every possible pair of numbers. If 1 is the *only* common divisor of a and b, then one says that a and b are *relatively prime*. For example, 9 and 16 are relatively prime.

Exercise 6 *Calculate the GCD of the following pairs of numbers:* 6 *and* 8*;* 6 *and* 18*;* 7 *and* 15*;* 30 *and* 20*.*

Common divisors are a very important idea in modular arithmetic; they are the key concept for understanding divisors of zero.

> **Fact.** If numbers x and n have a common divisor other than 1, then x is a divisor of zero modulo n.

For example, 2 is a common divisor of 6 and 8, and consequently 6 is a divisor of zero, modulo 8. Indeed, $6 \times 4 = 24 \equiv 0 \pmod 8$. It isn't difficult to explain this fact. Let $d > 1$ be a common divisor of x and n. By definition, $x \div d$ and $n \div d$ are both whole numbers, and therefore

$$x \times (n \div d) = (x \div d) \times n \equiv 0 \pmod n.$$

Exercise 7 *Write down the multiplication table modulo* 10*. Identify the units and the divisors of zero. Notice that* $10 = 2 \times 5$*. How is this fact relevant to your search for units and divisors of zero?*

2.4 The Euclidean Algorithm

It is easy enough to guess a GCD of two small numbers, but larger numbers require a more systematic approach. Fortunately there is a straightforward method, called the Euclidean algorithm, for calculating the GCD of two whole numbers. The algorithm is based on the following idea. Suppose the goal is to find the GCD of a pair of whole numbers x and y. It isn't hard to see that if a number d evenly divides both x and y, then d will also divide all of the following numbers: $x - y$, $x - 2y$, $x - 3y$, etc. Indeed d will evenly divide any number of the form $x - p \times y$. Conversely if d evenly divides y and $x - p \times y$, then d will also evenly divide x.

> **Conclusion:** for every whole number p, the GCD of x and y is equal to the GCD of y and $x - p \times y$.

For example, say one wants to compute the GCD of 168 and 91. One may as well be looking for the GCD of 91 and $168 - 91 = 77$. Repeating this reasoning, one should next look for the GCD of 77 and $91 - 77 = 14$, and then for the GCD of 14 and $77 - 5 \times 14 = 7$. Now $14 = 2 \times 7$, and therefore it is clear that 7 is the desired GCD. Indeed: $168 \div 7 = 24$ and $91 \div 7 = 13$. The algorithm is summarized below.

> **The Euclidean Algorithm.** Goal: to find the GCD of whole numbers $x > y$. If y divides x evenly then y is the GCD. Otherwise, calculate the remainder, r, from the division of x by y. In other words, $r = x - p \times y$ where p is the whole part of $x \div y$. Since GCD(x, y)=GCD(y, r), one goes back to the beginning of the algorithm and restarts the calculation with the new x equal to the old y, and the new y equal to the old r. Since $y < x$ and $r < y$, the inputs to the algorithm become smaller with each iteration. The algorithm is, therefore, guaranteed to terminate after a finite number of repetitions.

Exercise 8 *Use the Euclidean algorithm to find the GCD of* 1113 *and* 504.

It is easy enough to see that if d evenly divides a pair of numbers x and y, then d will also evenly divide $a \times x + b \times y$ for all integers a and b. The following fact is a little more surprising.

> **Fact.** Given whole numbers x and y, one can find integers a and b so that $a \times x + b \times y$ is exactly equal to the GCD of x and of y. In particular, if x and y are relatively prime, then one can find a and b so that $a \times x + b \times y = 1$.

This fact is enormously useful in connection with the calculation of reciprocals in modular arithmetic; this point will be explained shortly. First, however, one needs to master a modified version of the Euclidean algorithm, a version tailored to the calculation of the critical a and b. Proceeding by way of example, suppose one is interested in the GCD of 1113 and 504. Consider the following table.

n	p	a	b
1113		1	0
504	2	0	1
105	4	1	-2
84	1	-4	9
21	4	5	-11
0			

In what follows, subscripts are used to specify the number of the row. Thus n_3 refers to 105, the n entry in the third row. The first column contains the sequence of numbers obtained in the course of applying the Euclidean algorithm (c.f. Exercise 8). In other words, n_3 is the remainder of $n_1 \div n_2$, n_4 is the remainder of $n_2 \div n_3$, etc. The p column contains the whole part of each of these divisions: p_2 is the whole part of $n_1 \div n_2$, p_3 is the whole part of $n_2 \div n_3$, etc. In other words,

$$\begin{aligned} n_3 &= n_1 - p_2 \times n_2, \\ n_4 &= n_2 - p_3 \times n_3, \\ n_5 &= n_3 - p_4 \times n_4, \end{aligned}$$

The entries in the a and b columns are chosen so that $n = a \times 1113 + b \times 504$ in each row of the table. For example, row four contains $84 = -4 \times 1113 + 9 \times 504$. The final row (not counting the last zero) contains the final, desired a and b. Indeed, $21 = 5 \times 1113 - 11 \times 504$. For obvious reasons, $a = 1$, $b = 0$ in the first row, and $a = 0$, $b = 1$ in the second row. For the subsequent rows the entries are calculated in the same fashion as the entries in the n column. To be more specific:

$$\begin{aligned} a_3 &= a_1 - p_2 \times a_2, \\ a_4 &= a_2 - p_3 \times a_3, \\ a_5 &= a_3 - p_4 \times a_4, \end{aligned}$$

and so on. The b column is computed in the same way. I will leave it to you to figure out why this method ensures that $n = a \times 1113 + b \times 504$ in every row.

Exercise 9 *Calculate the GCD of* 1466 *and* 237*, and find the integers a, b such that $a \times 1466 + b \times 237$ is equal to the GCD.*

You are now in possession of a method that makes it easy to calculate reciprocals in modular arithmetic. Remember, that x is a unit modulo n if and only if x and n are relatively prime. If this is the case, then one simply finds a and b so that $a \times x + b \times n = 1$, and voila: $a \times x \equiv 1 \pmod{n}$, i.e. $a \equiv x^{-1} \pmod{n}$.

Exercise 10 *Calculate* $59 \div 237$ *modulo* 1466*. Check your answer.*

2.5 Curiouser and curiouser ....

Presented for your reading pleasure, the following curious tidbit from the world of recreational mathematics. Take out a calculator and randomly punch in a whole number. Note the last digit of your chosen number, and then raise your number to the fifth power. If your starting number wasn't too large, and if your calculator has sufficiently many digits on its display, then you will notice that the last digit of the result is the same as the last digit of the starting number. For example: $17^5 = 1419857$ and $22^5 = 5153632$. This interesting phenomenon occurs for other powers as well: the powers $1, 5, 9, 13, 17, 21$, etc, will all work. For example, $7^9 = 40353607$ and $2^{13} = 8192$.

You may have realized that the business of looking at the last digit can be handled most conveniently in terms of modular arithmetic. The fact is that modulo 10, a number is equivalent to its last digit, and therefore this curious fact can be written down as the following identity:

$$x^p \equiv x \pmod{10}, \quad \text{as long as } p = 1, 5, 9, 13, 17, 21, 25, \ldots$$

The above identity is far from being a mere curiosity. It is, in fact, the essential component of the RSA encryption/decryption process. The application to cryptography will be explained in the next section. First, it will be necessary to take a closer look at the above "curious fact", and try to find the analogous trick for moduli other than 10.

As you may have noticed, the critical exponents, i.e. the values of p for which $x^p \equiv x \pmod{10}$, are precisely the numbers p such that $p-1$ is a multiple of 4. To put it differently, the list of critical exponents is obtained by starting with 1 and then repeatedly adding 4. Why the number 4 though? Look back at Exercise 7, and *count the number of units* in that particular system of modular arithmetic. That's right; there are 4 units, and for reasons that we won't get into here, the spacing in the list of critical exponents is always equal to the total number of units in one's chosen system of modular arithmetic. This all-important principle is summarized below.

> **A Curious Fact.** Suppose that n is a square free whole number, and let ϕ be the total number of units in the system of arithmetic modulo n. In such circumstances the following identity holds: $x^p \equiv x \pmod{n}$ where the exponent can be any of the critical values $p = 1, 1+\phi, 1+2\phi, 1+3\phi, \ldots$

The business about square-free numbers is an important, but technical detail. The problem is that the above "curious fact" does not work for certain values of the modulus n. The values of n for which the trick fails to work are precisely those whole numbers that can be divided evenly by a square. For example, working modulo 8 and starting with 2 one has

$$2^2 \equiv 4, \quad 2^3 \equiv 0, \quad 2^4 \equiv 0, \quad 2^5 \equiv 0, \quad \text{etc}$$

and so one never gets back to 2, no matter which power is used. In other words, the trick doesn't work when 8 is the modulus. Now 8 can be divided by 4, and the latter is a square, and that is why the trick fails to work modulo 8. The bottom line is that the "curious fact" will work if and only if one chooses a clock size (i.e. a modulus n) that does not have any squares as a factor.

Exercise 11 *What are the critical exponents p in the system of arithmetic modulo 22? Do some calculations and verify that* $x^p \equiv x \pmod{22}$ *for these critical values of p.*

Exercise 12 *Explain why, in the modulo 22 system, it suffices to consider only the exponents from 1 to 10. Calculate* $3^{32} \pmod{22}$ *just "by looking at it".*

3 RSA

Those of you who navigate the World Wide Web using Netscape browsers may have noticed that the very first screen shown by Navigator and Communicator has a bunch of corporate looking logos, including one that has a pair of keys alongside the letters RSA. The icon in question is the logo of **RSA Data Security**, the owners of the patented RSA public key encryption algorithm. The letters RSA stand for Rivest, Shamir and Adleman; these are the names of the M.I.T. (Massachusetts Institute of Technology) academics who invented the algorithm in 1977. In 1982 these three individuals went on to found **RSA Data Security**, a company that specializes in secure digital communication. Another significant acronym in the world of computer cryptography is **PGP**, which stands for *pretty good privacy.* This is a loosely organized collection of (mostly) free software that puts the power of private and secure digital communication in the hands of ordinary citizens. PGP relies on the RSA algorithm for its core functionality.

3.1 Public key cryptography.

What is the basic goal of cryptography? Speaking generally, one wants a method for encoding human-readable messages into an unreadable cipher, as well as a corresponding method for translating the coded information back into everyday language. Typically, a cryptographic method involves some sort of a secret key. One simple example is the so-called substitution cipher. One assigns a number to each letter of the alphabet, and then encodes and decodes messages using this code. In theory, people who are not privy to the translation table of letter-number pairs, should not be able to decode and understand the message (A skilled codebreaker, however, can break a substitution cipher without much trouble.)

The substitution cipher is a basic example of traditional, single key cryptography. The reason for the name “single key” (“secret key” is also used) is that both the sender and receiver of an encoded message make use of the same, secret key — in this case the table of letter-number pairs — to generate and decipher the message. All single key encryption schemes share a fundamental difficulty: the encryption key must be agreed on beforehand and then kept absolutely secret. Thus, single key encryption is both inconvenient (one has to find a secure means of sharing the encryption key between the communicating parties) and fundamentally insecure in a group setting (

it takes just one set of “loose lips” to blow the security of a single-key code that is shared by a group of individuals).

Public key cryptography is based on a very simple, very beautiful idea that effectively deals with both of the above issues. The idea is that everybody should possess two encryption keys: a public key, and a private key. Furthermore, the encryption method should work so that messages encoded using person X’s **public** key can only be decoded using person X’s **private** key. Likewise a message encoded with a private key should only be decodable using the corresponding public key. Next, everyone who wants to communicate shares their public keys, but keeps their private key strictly hidden. With such an arrangement, in order to communicate with person X, the sender will encode a message using person X’s **public key**. Now everyone in the world knows person X’s public key, but it doesn’t do them any good, because it will take person X’s **private key** to actually decode the message.

A system of public key cryptography can also be used to establish identity, and to enable secure financial transactions. Let’s say that I show up at a bank and claim that I am person X. If I am indeed person X, then I should be able to encode a simple message, something to the effect of “Hello, my name is X.”, using my secret, private key. Now everyone, including the bank, has access to person X’s **public** key, and so should be able to decipher the simple message, and verify that it really was encrypted using person X’s **private** key. The point is that person X, and *only person X*, could have created a message that is decodable using person X’s public key. In this way an arrangement of public and private keys can be made to serve as a secure authentication system (very useful for bank and credit card transactions, as you may well imagine).

Credit for the invention of public key cryptography is typically given to Diffie and Hillman: two computer scientists who publicized the first public key method back in 1975.

3.2 Modular arithmetic to the rescue

The next issue to consider is the means by which one can implement such a scheme of public and private key encryption. If you’ve read the preceding section on modular arithmetic, then you already have in your possession all of the mental equipment required for such a task.

First, recall the following rule of high school algebra:

$$(a^b)^c = a^{bc}.$$

There is nothing mysterious about this rule. As a particular example think of the rule with $b = 2$ and $c = 3$. In this instance the rule is saying that if one squares a number a and then cubes the result, the final answer will be a raised to the sixth power. Or to put it another way:

$$(a^2)^3 = (a^2) \times (a^2) \times (a^2) = a \times a \times a \times a \times a \times a = a^6.$$

Now in modular arithmetic, there are certain powers that do absolutely nothing; this is the "curious fact" discussed in Section 2.5. For example, in Exercise 11 we saw that $x^{21} \equiv x \pmod{22}$. Note that $21 = 3 \times 7$. It therefore stands to reason that working modulo 22, if I first raise a number to the power 3, and then raise the result to the power 7, the final result will be the number that I started with. Eureka! Why don't I then use the number 3 as a private key, the number 7 as the public key, and encrypt messages by raising numbers to these powers in modulo 22 arithmetic? The following calculation illustrates this. Say you wants to send me a message that consists of the numbers: $2, 3, 8$. You raise each of these numbers to the power 7 — that being my public key — and so the message I receive is $18, 9, 2$, because

$$2^7 = 128 \equiv 18, \quad 3^7 = 2187 \equiv 9, \quad 8^7 = 2,097,152 \equiv 2,$$

where all equivalences are $\pmod{22}$. I then take the message, raise each number to the power 3, and obtain

$$18^3 = 5832 \equiv 2, \quad 9^3 = 729 \equiv 3, \quad 2^3 \equiv 8;$$

I get back the original message! That's all there is to it; the above example illustrates the essentials of the RSA public key encryption system.

Exercise 13 *Construct a table that shows the effect of raising numbers in the modulo* 22 *system to the powers* 3 *and* 7*. Inspect this table and confirm that, indeed, the two operations undo one another.*

3.3 Some final details.

The above discussion of the RSA algorithm has not, as yet, addressed the following simple, but crucial question: how does one know that the RSA public key encryption method is secure? The fact of the matter is that the method *is not secure* unless one is careful about the way one chooses the modulus, n. Consider again the example of the preceding section: it is public knowledge that $n = 22$ and that $e = 7$, but the other exponent, f, is not publicized. A moment of reflection suffices to show that knowledge of n and e allows one to guess the supposedly secret value of f. Indeed, knowing that $n = 22$, one also knows that $\phi = 10$ (see Exercise 11), and hence that whatever f is, $e \times f$ must be one of the following numbers: $1, 11, 21, 31, 41, \ldots$ In other words, f, whatever it is, must satisfy the equation $7 \times f \equiv 1 \pmod{10}$. A quick check of the numbers from 1 to 9 will reveal that $f = 3$ is a solution to the above equation. The supposedly secret, private key is thereby revealed, and the entire method of encryption rendered useless.

The point of the above example, is that one has to be careful about choosing the modulus n. Indeed, part of the RSA methodology, something that has not been mentioned so far, is a certain procedure for choosing n that makes the subsequent process of encryption/decryption secure. According to RSA, one begins by choosing two prime numbers, call them p and q, and sets $n = p \times q$ (Recall that a number is called prime if it has no divisors except for 1 and itself.) An added advantage of choosing n in this manner is the ease with which one can then calculate ϕ, the number of units modulo n.

Fact. If n is the product of primes p and q, then $\phi = (p-1)(q-1)$.

This formula is confirmed by the examples considered so far. Look back at Exercise 7. The modulus was $10 = 2 \times 5$, and the number of units was $4 = (2 - 1) \times (5 - 1)$; just as predicted by the formula. In Exercise 11 n was $22 = 2 \times 11$, and the modulus was $10 = (2 - 1) \times (11 - 1)$, also in agreement with the formula.

I won't prove the above formula for ϕ in these notes, but rather indicate why the formula is valid by considering a particular example. Let us consider $n = 15 = 3 \times 5$. Arrange the numbers in the modulo 15 system in a three by five rectangular array like so:

	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

What is the meaning of the above arrangement? Every number x in the modulo 15 system can be described equally well by the following two items of information: the value of x modulo 3, and by the value of x modulo 5. Look, for example in row 1, column 3 of the above table, and you will find the number 13. The reason for this placement is that $13 \equiv 1 \pmod 3$ and that $13 \equiv 3 \pmod 5$. Similarly $12 \equiv 0 \pmod 3$ and $12 \equiv 2 \pmod 5$, and therefore 12 is located in row 0, column 2.

Having understood the arrangement you will see that the numbers that are divisible by 3 are located in row 0, while the numbers that are divisible by 5 are located in column 0. Furthermore, the divisors of zero in the system modulo 15 are the numbers that are divisible either by 3 or by 5 (I leave it to you to figure out the reason for this.) Therefore the units of the system are going to be all the numbers outside row 0 and column 0 of the above table. In other words, the units are the numbers found in the two by four rectangle formed by rows 1 and 2, and columns 1 through 4. No wonder then that there is a total of $2 \times 4 = 8$ units.

Returning to considerations of security, one needs to choose the n so that the corresponding ϕ is enormously difficult to discover. This is accomplished by using very large primes p and q; it is not unusual to take p and q to be a hundred digits long! Now multiplying two numbers a hundred digits long will produce a number two hundred digits long — a tedious process but one that a computer can handle easily. However, factoring a such a large number into its prime components is a task that cannot be accomplished even by today's fastest computers. Factorization, is fundamentally a process of trial and error, and with numbers that large, there are just too many possibilities to check.

The upshot is that with very large p and q, one can publicize n, and the values of p and q will remain safely uncomputable. Since the knowledge of p and q remains secret, there is no way for an outside agency to calculate ϕ, and therefore no way to guess the private key f, either. The cryptosystem can then be considered to be secure.

It is important to make one final remark regarding computations in modular arithmetic. At first glance, the calculation of powers when the modulus is large poses a considerable computational challenge. Say that one wanted to calculate $48^{29} \pmod{221}$. Ostensibly,

$$\begin{aligned}48^{29} &= 5,701,588,684,667,867,878,541,238,858,441,350,344,816,132,620,288\\ &= 2,579,904,382,202,655,148,661,194,053,593,371,196,749,381,276 \times 221 + 107.\end{aligned}$$

In other words, this gargantuan power is equivalent to 107 $(\text{mod}\, 221)$. This sort of brute-force calculation is beyond the capability of most hand-held calculators. Fortunately there is a more elegant way to do the calculation. Remember that 48^{29} is obtained by multiplying 48 together 29 times. The trick is to reduce modulo 221 at the intermediate stages of the calculation, rather than waiting to reduce at the very end. Note that $29 = 6 \times 4 + 5$, and hence that

$$48^{29} = (48^6)^4 \times 48^5.$$

The overall calculation can therefore be done like this (all equivalences are modulo 221):

$$48^6 = 12,230,590,464 \equiv 66, \quad (48^6)^4 \equiv 66^4 = 18,974,736 \equiv 118,$$

$$48^5 = 254,803,968 \equiv 29, \quad (48^6)^4 \times 48^3 \equiv 118 \times 29 = 3,422 \equiv 107.$$

There are many ways to break up this calculation. It all depends on how one chooses to break up the exponent 29.

Exercise 14 *Calculate* 29^{48} *modulo* 221.

3.4 A step by step description of the RSA algorithm.

- Choose prime numbers p and q, and set $n = p \times q$. The larger the values of p and q, the more secure the resulting encryption system.

- Let ϕ denote the total number of units in the corresponding system of modular arithmetic. Having chosen n to be the product of primes p and q, one has $\phi = (p-1)(q-1)$.

- Next, one chooses two whole numbers e and f in such a way that $e \times f$ is equal to one of the critical exponents, i.e. to one of the following numbers: $1, 1+\phi, 1+2\phi, 1+3\phi, \ldots$. This is done by first, choosing $e < \phi$ so that e and ϕ are relatively prime; second, calculating the reciprocal of e modulo ϕ using the modified Euclidean algorithm; and finally setting f equal to that reciprocal. A useful trick is to use a prime number less than ϕ as e. This way one is certain that e and ϕ are relatively prime, and hence that e is a unit modulo ϕ.

- One tells the whole world that one's public key consists of the modulus n and the exponent e. The exponent f is kept a secret.
- Whenever someone wants to send a message — and let us suppose that the message consists of a string of numbers modulo n — the sender will raise each number in the message to the power e. To decode the original message, one raises each of the encoded numbers to the power f.
- Conversely, to establish one's identity, one would compose a signature message, and raise the numbers in the message to the power f. The receiver would then perform the authentication by raising the numbers in the encoded message to the power e, and thereby decode the original signature message.

3.5 A final example.

The present section illustrates the RSA algorithm with a simple example. First, one needs to choose a pair of primes — for instance, $p = 13$ and $q = 17$. From there, $n = 13 \times 17 = 221$, and $\phi = 12 \times 16 = 192$.

Next, one needs to choose a public key e and a private key f in such a way that $e \times f \equiv 1 \pmod{192}$. For this example let $e = 29$; this is a prime number, and therefore guaranteed to be a unit modulo any n. Using the modified Euclidean algorithm one calculates that $-8 \times 192 + 53 \times 29 = 1$. Hence $f = 53$, because $23 \times 53 \equiv 1 \pmod{192}$.

In order to send messages there must be a standard way to encode letters in terms of numbers. For the purposes of this exercise, the following encoding will be used:

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z	space
14	15	16	17	18	19	20	21	22	23	24	25	26	27

Thus, the message "HELLO" corresponds to the following string of numbers: 8, 5, 5, 12, 16. Encoding the message using the public key means raising each of these numbers to the power 29 modulo 221. The result is the following encoded message: $60, 122, 122, 116, 152$. To decode the message using the private key, one raises each of these numbers to the 53^{rd} power modulo 221, and recovers $8, 5, 5, 12, 16$ — the original message!

4 Appendix. Answers to exercises.

Exercise 1. T, T, F, T. Note that $9+9+9 = 27 = 1+2\times 13 \equiv 1 \pmod{13}$.

Exercise 2.

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Exercise 3. $3 \times 2 \equiv 1 \pmod 5$, and hence $3^{-1} \equiv 2 \pmod 5$. Similarly, $4 \times 4 \equiv 1 \pmod 5$, and therefore $4^{-1} \equiv 4 \pmod 5$. Regarding the division problems one has: $4 \div 3 \equiv 3$, $\quad 3 \div 4 \equiv 2$, $\quad 3 \div 3 \equiv 1$, $\quad 1 \div 3 \equiv 2$, where all equivalences are $\pmod 5$.

Exercise 4.

$\times$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Exercise 5. If a number a is a unit then, by definition, there is a number b such that $b \times a \equiv 1$. Therefore if $c \neq 0$, then $a \times c$ cannot possibly be zero either, because $b \times a \times c \equiv 1 \times c \equiv c$. On the other hand, if a is a divisor of zero, then $a \times b \equiv 0$ for some non-zero b. Therefore it is useless to look for a number c such that $c \times a \equiv 1$. The reason is that if such a c were to exist then $c \times a \times b$ would be equal to $1 \times b \equiv b$. However we know that $c \times a \times b \equiv 0$.

Exercise 7

$\times$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	0	2	4	6	8
3	3	6	9	2	5	8	1	4	7
4	4	8	2	6	0	4	8	2	6
5	5	0	5	0	5	0	5	0	5
6	6	2	8	4	0	6	2	8	4
7	7	4	1	8	5	2	9	6	3
8	8	6	4	2	0	8	6	4	2
9	9	8	7	6	5	4	3	2	1

The divisors of zero are 2, 4, 6, 8, and 5. These are the numbers that are divisible either by 2 or by 5. All the other numbers are units: 1, 3, 7, 9.

Exercise 6 Answers: 2, 6, 1, 10.

Exercise 8

$$1113 = 2\times504+105, \quad 504 = 4\times105+84, \quad 105 = 1\times84+21, \quad 84 = 4\times21+0.$$

Therefore 21 is the GCD of 1113 and 504.

Exercise 9

n	p	a	b
1466		1	0
237	6	0	1
44	5	1	-6
17	2	-5	31
10	1	11	-68
7	1	-16	99
3	2	27	-167
1	3	-70	433

Answer: the GCD is 1. It is given by $1 = -70 \times 1466 + 433 \times 237$.

Exercise 10 From the preceding exercise we know that $433\times237 \equiv 1 \pmod{1466}$, and hence that $237^{-1} \equiv 433 \pmod{1466}$. Therefore

$$59 \div 237 \equiv 59 \times 433 = 25,547 \equiv 625 \pmod{1466}.$$

Checking this answer:

$$625 \times 237 = 148,125 = 101 \times 1466 + 59 \equiv 59 \pmod{1466}.$$

Exercise 11 Note that $22 = 2 \times 11$. Therefore, the divisors of zero in the mod 22 system are going to be the numbers that are divisible either by 2 or by 11; i.e. $2, 4, 6, 8, 10, 12, 14, 16, 18, 20$, and 11 — a total of eleven divisors of zero, twelve if one includes 0 itself. That leaves a total of $22 - 12 = 10$ units, i.e. $\phi = 10$, and therefore the critical exponents are going to be $p = 1, 11, 21, 31, 41, 51, \ldots$.

Consider a computation with $p = 11$ and $x = 7$. Now $7^{11} = 1,977,326,743$. What is this number modulo 22? To get the answer one must divide by 22 and calculate the remainder. The division gives $89,878,488$. As for the remainder, it is $7^{11} - 89,878,488 \times 22 = 7$, the starting number. Try it again with $p = 21$ and $x = 3$. One gets $3^{21} = 10,460,353,203$. Dividing the latter by 22 one gets $475,470,600$ plus a remainder. To calculate the remainder one does $3^{21} - 475,470,600 \times 22$; the answer is 3, just as expected.

Exercise 12 It is evident that in the modulo 22 system, exponents that differ by a multiple of 10 end up giving the same result. Here is why. We already noted that

$$\ldots \equiv x^{31} \equiv x^{21} \equiv x^{11} \equiv x^{1} \pmod{22}.$$

Multiplying through by x one observes that

$$\ldots \equiv x^{32} \equiv x^{22} \equiv x^{12} \equiv x^{2} \pmod{22},$$

and of course one can keep going to conclude that

$$\ldots \equiv x^{37} \equiv x^{27} \equiv x^{17} \equiv x^{7} \pmod{22},$$

and so on and so forth. In conclusion, if one starts with an exponent that is greater than 10 (such as 32 for example) one can subtract a multiple of 10 from the exponent without affecting the answer. In particular, $3^{32} \equiv 3^2 \equiv 9 \pmod{22}$.

Exercise 13

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
x^3	1	8	5	20	15	18	13	6	3	10	11	12	19	16	9	4	7	2	17	14	21
x^7	1	18	9	16	3	8	17	2	15	10	11	12	7	20	5	14	19	6	13	4	21

Exercise 14 The answer is 1. Note that $48 = 6 \times 4 \times 2$, and hence that $29^{48} = ((29^6)^4)^2$. From there

$$29^6 = 594,823,321 \equiv 53, \quad 53^4 = 7,890,481 \equiv 118,$$

$$29^{48} = ((29^6)^4)^2 \equiv 118^2 = 13,924 \equiv 1,$$

where all equivalences are, of course, modulo 221.